Non-symplectic symmetries and bi-Hamiltonian structures of the rational harmonic oscillator

2002 J. Phys. A: Math. Gen. 35 L679

(http://iopscience.iop.org/0305-4470/35/47/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L679–L686 PII: S0305-4470(02)54325-0

LETTER TO THE EDITOR

Non-symplectic symmetries and bi-Hamiltonian structures of the rational harmonic oscillator

José F Cariñena[1,4], **Giuseppe Marmo**[2,3] **and Manuel F Rañada**[1]

[1] Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
[2] Dipartimento di Scienze Fisiche, Universitá Federico II di Napoli, Napoli, Italy
[3] INFN, Sezione di Napoli, Complesso Univ. di Monte Sant'Angelo, Via Cintia, 80125 Napoli, Italy

E-mail: jfc@posta.unizar.es, giuseppe.marmo@na.infn.it and mfran@posta.unizar.es

Received 2 October 2002
Published 12 November 2002
Online at stacks.iop.org/JPhysA/35/L679

Abstract
The existence of bi-Hamiltonian structures for the rational harmonic oscillator (non-central harmonic oscillator with rational ratio of frequencies) is analysed by making use of the geometric theory of symmetries. We prove that these additional structures are a consequence of the existence of dynamical symmetries of non-symplectic (non-canonical) type. The associated recursion operators are also obtained.

PACS numbers: 02.30.Ik, 42.20.Jj, 02.40.Yy
Mathematics Subject Classification: 37J15, 37J35, 70H33

1. Non-symplectic symmetries

It is well known that there is a close relation [1] between integrability and the existence of alternative structures (see e.g. [2] for a recent paper) and also that integrable systems are systems endowed with a large number of symmetries. The purpose of this letter is to analyse, in the particular case of the $n = 2$ harmonic oscillator, how these additional structures arise from the existence of dynamical symmetries of non-symplectic (non-canonical) type.

Let (M, ω_0, H) be a Hamiltonian system and $\Gamma_{\rm H}$ the associated Hamiltonian vector field, defined by $i(\Gamma_{\rm H})\omega_0 = {\rm d}H$. A (infinitesimal) dynamical symmetry of this system is a vector field $Y \in \mathfrak{X}(M)$ such that $[Y, \Gamma_{\rm H}] = 0$. When Y is a dynamical but non-symplectic symmetry of the system, then we have that (i) the dynamical vector field $\Gamma_{\rm H}$ is bi-Hamiltonian, and (ii) the function $Y(H)$ is the new Hamiltonian, and therefore is a constant of motion.

A sketch of the proof [3–6] of this statement is as follows: The vector field Y does not preserve ω_0 and, as it is a non-canonical transformation, it determines a new 2-form

[4] Author for correspondence.

$\omega_Y = \mathcal{L}_Y\omega_0$ ($\mathcal{L}_Y$ denotes the Lie derivative with respect to Y). As Y is a symmetry, $[Y, \Gamma_{\rm H}] = 0$, then $\mathcal{L}_Y \circ i_{\Gamma_{\rm H}} = i_{\Gamma_{\rm H}} \circ \mathcal{L}_Y$, and, consequently,

$$i_{\Gamma_{\rm H}}\omega_Y = i_{\Gamma_{\rm H}}\mathcal{L}_Y\omega_0 = \mathcal{L}_Y i_{\Gamma_{\rm H}}\omega_0 = \mathcal{L}_Y(\mathrm{d}H) = \mathrm{d}(YH).$$

Therefore, the 2-form ω_Y is admissible for the dynamical vector field $\Gamma_{\rm H}$, i.e. $\mathcal{L}_{\Gamma_{\rm H}}\omega_Y = 0$, which is weakly bi-Hamiltonian with respect to the original symplectic 2-form ω_0 and the new structure ω_Y. Of course, the particular form of ω_Y depends on Y and, in some cases, it can be just a constant multiple of ω_0 (trivial bi-Hamiltonian system). In some other cases ω_Y may be a degenerate 2-form with a nontrivial kernel. In any case, the vector field $\Gamma_{\rm H}$ is a dynamical system solution of the following two equations:

$$i(\Gamma_{\rm H})\omega_0 = \mathrm{d}H \quad \text{and} \quad i(\Gamma_{\rm H})\omega_Y = \mathrm{d}[Y(H)]$$

Therefore, the function $H_Y = Y(H)$, which must be a constant of motion, can be considered as a new Hamiltonian for $\Gamma_{\rm H}$.

2. Bi-Hamiltonian structures of the rational harmonic oscillator

The two-dimensional harmonic oscillator

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + \tfrac{1}{2}\left(\lambda_1^2 x^2 + \lambda_2^2 y^2\right) \tag{1}$$

has the two one-degree-of-freedom energies, $I_1 = E_x$ and $I_2 = E_y$, as fundamental constants of motion. The superintegrability of the rational case, $\lambda_1 = m\lambda_0, \lambda_2 = n\lambda_0$, with $m, n \in \mathbb{N}$, can be proved by making use of a complex formalism [7, 8]. Let K_x, K_y be the following two functions $K_x = p_x + \mathrm{i}m\lambda_0 x$ and $K_y = p_y + \mathrm{i}n\lambda_0 y$; then the Hamiltonian H and the canonical symplectic form ω_0 become

$$H = \tfrac{1}{2}(K_x K_x^* + K_y K_y^*)$$

and

$$\omega_0 = \frac{\mathrm{i}}{2m\lambda_0}\,\mathrm{d}K_x \wedge \mathrm{d}K_x^* + \frac{\mathrm{i}}{2n\lambda_0}\,\mathrm{d}K_y \wedge \mathrm{d}K_y^*.$$

We have

$$\{K_x, K_x^*\} = 2\,\mathrm{i}m\lambda_0 \qquad \{K_y, K_y^*\} = 2\,\mathrm{i}n\lambda_0$$

and, therefore, the evolution equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}K_x = \mathrm{i}m\lambda_0 K_x \qquad \frac{\mathrm{d}}{\mathrm{d}t}K_y^* = -\mathrm{i}n\lambda_0 K_y^*.$$

Hence, the complex function J defined as

$$J = K_x^n (K_y^*)^m \tag{2}$$

is a constant of motion that determines two different real first integrals, $I_3 = \mathrm{Im}(J)$ and $I_4 = \mathrm{Re}(J)$, which are polynomials in the momenta of degree $m + n - 1$ and $m + n$, respectively. As an example, for the isotropic case, $\lambda_1 = \lambda_2 = \lambda_0$, we obtain

$$\mathrm{Re}(J) = p_x p_y + \lambda_0^2 xy \qquad \mathrm{Im}(J) = \lambda_0(xp_y - yp_x).$$

$\mathrm{Im}(J)$ is just the angular momentum and $\mathrm{Re}(J)$ is the non-diagonal component of the Fradkin tensor [9]. For the first non-isotropic case, $\lambda_1 = \lambda_0$, $\lambda_2 = 2\lambda_0$, we obtain

$$\mathrm{Re}(J) = p_x^2 p_y + \lambda_0^2(4yp_x - xp_y)x \qquad \mathrm{Im}(J) = (xp_y - yp_x)p_x + \lambda_0^2 x^2 y.$$

This complex procedure provides not only the fundamental constant I_3, but the pair (I_3, I_4); although the 'partner' function I_4 is not independent (it is a function of I_1, I_2, I_3), we will see that it plays an important role, since it is closely concerned with the bi-Hamiltonian formalism. In fact, we will take the complex function J as our starting point for the search of symmetries, but J means not only one but two functions, I_3 and I_4.

Noether theorem in the Hamiltonian formalism states that all constants of motion arise from canonical symmetries of the Hamiltonian function. Moreover, in differential geometric terms, the infinitesimal symmetries are simply those corresponding to the Hamiltonian vector fields, with respect to the canonical structure ω_0, defined by the constants of motion. In this particular case, the above complex function J given by (2) arises from a symmetry of (1) represented by the complex vector field X_J defined by

$$i(X_J)\omega_0 = \mathrm{d}J \qquad X_J(H) = 0. \tag{3}$$

In the following, and for easy of notation, we will suppose $\lambda_0 = 1$.

Proposition 1. *The complex vector field X_J, defined by (3) as the canonical infinitesimal symmetry associated with J, can be written as a linear combination of two dynamical but non-symplectic symmetries of* Γ_{H}.

Proof. Let us denote by Y_{xm} and Y_{yn} the Hamiltonian vector fields of K_x and K_y

$$i(Y_{xm})\omega_0 = \mathrm{d}K_x \qquad i(Y_{yn})\omega_0 = \mathrm{d}K_y$$

with coordinate expressions

$$Y_{xm} = \frac{\partial}{\partial x} - \mathrm{i}m\frac{\partial}{\partial p_x} \qquad Y_{yn} = \frac{\partial}{\partial y} - \mathrm{i}n\frac{\partial}{\partial p_y}.$$

Note that, as $H = I_1 + I_2$ with $|K_x|^2 = 2I_1$ and $|K_y|^2 = 2I_2$, we have

$$\Gamma_{\mathrm{H}} = \mathrm{Re}(K_x^* Y_{xm} + K_y Y_{yn}^*).$$

Then, the complex vector field X_J, canonical infinitesimal symmetry of the harmonic oscillator, can be written as the following linear combination

$$X_J = nY + mY'$$

where Y, Y' are given by

$$Y = \left(K_x^{(n-1)} K_y^{*m}\right) Y_{xm} \qquad Y' = \left(K_x^n K_y^{*(m-1)}\right) Y_{yn}^*.$$

The important point is that these two vector fields, Y and Y', are neither locally Hamiltonian with respect to ω_0

$$\mathcal{L}_Y\omega_0 \neq 0 \qquad \mathcal{L}_{Y'}\omega_0 \neq 0$$

nor infinitesimal symmetries of the Hamiltonian

$$\mathcal{L}_Y H \neq 0 \qquad \mathcal{L}_{Y'} H \neq 0.$$

Concerning the Lie bracket of Y with the dynamical vector field Γ_{H}, it is given by

$$[Y, \Gamma_{\mathrm{H}}] = \left(K_x^{n-1} K_y^{*m}\right)[Y_{xm}, \Gamma_{\mathrm{H}}] - \Gamma_{\mathrm{H}}\left(K_x^{n-1} K_y^{*m}\right) Y_{xm}$$

but as

$$[Y_{xm}, \Gamma_{\mathrm{H}}] = -\mathrm{i}Y_{xm} \qquad [Y_{yn}^*, \Gamma_{\mathrm{H}}] = -\mathrm{i}Y_{yn}^*$$

and

$$\begin{aligned}\Gamma_{\mathrm{H}}\left(K_x^{n-1} K_y^{*m}\right) &= (n-1)(\mathrm{i}m)\left(K_x^{n-1} K_y^{*m}\right) + m(-\mathrm{i}n)\left(K_x^{n-1} K_y^{*m}\right) \\ &= -\mathrm{i}m\left(K_x^{n-1} K_y^{*m}\right)\end{aligned}$$

we obtain

$$[Y, \Gamma_{\rm H}] = 0.$$

Thus, Y is a dynamical but non-symplectic (non-canonical) symmetry of $\Gamma_{\rm H}$. It can be proved, in a similar way, that this property is also true for Y'. Note that, in the language of 1-forms, this property arises from the fact that $\mathrm{d}J$ splits as a sum of two non-closed 1-forms that, nevertheless, remain invariant under $\Gamma_{\rm H}$, that is, $\mathrm{d}J = n\phi_1 + m\phi_2$, $\mathrm{d}\phi_r \neq 0$, $\mathcal{L}_{\Gamma_{\rm H}}(\phi_r) = 0$, $r = 1, 2$. □

Two new structures can be obtained from ω_0 by Lie derivation with respect to Y and Y'. If we denote by ω_Y and ω'_Y these two new 2-forms, $\omega_Y = \mathcal{L}_Y\omega_0$ and $\omega'_Y = \mathcal{L}_{Y'}\omega_0$, then we obtain

$$\omega_Y = -m\big(K_x^{(n-1)}K_y^{*(m-1)}\big)\,\mathrm{d}K_x\wedge\mathrm{d}K_y^* \qquad \omega'_Y = n\big(K_x^{(n-1)}K_y^{*(m-1)}\big)\,\mathrm{d}K_x\wedge\mathrm{d}K_y^*.$$

In the following we will denote by Ω the complex 2-form defined as

$$\Omega = \mathrm{d}K_x\wedge\mathrm{d}K_y^* = \Omega_1 + \mathrm{i}\Omega_2$$

where the two real 2-forms, $\Omega_1 = \mathrm{Re}(\Omega)$ and $\Omega_2 = \mathrm{Im}(\Omega)$, take the form

$$\Omega_1 = mn\,\mathrm{d}x\wedge\mathrm{d}y + \mathrm{d}p_x\wedge\mathrm{d}p_y \qquad \Omega_2 = m\,\mathrm{d}x\wedge\mathrm{d}p_y + n\,\mathrm{d}y\wedge\mathrm{d}p_x.$$

Note that ω_Y and ω'_Y satisfy the relation $n\omega_Y + m\omega'_Y = 0$. Actually, this can be considered as a consequence of the fact that X_J is locally Hamiltonian with respect to the canonical form ω_0.

Proposition 2. *The dynamical vector field $\Gamma_{\rm H}$ of the rational harmonic oscillator is a bi-Hamiltonian system with respect to (ω_0, ω_Y).*

Proof. Note that

$$i(\Gamma_{\rm H})\omega_Y = -m\big(K_x^{(n-1)}K_y^{*(m-1)}\big)i(\Gamma_{\rm H})\Omega$$

and as

$$i(\Gamma_{\rm H})\Omega = \Gamma_{\rm H}(K_x)\,\mathrm{d}K_y^* - \Gamma_{\rm H}(K_y^*)\,\mathrm{d}K_x = \mathrm{i}mK_x\,\mathrm{d}K_y^* + \mathrm{i}nK_y^*\,\mathrm{d}K_x$$

we obtain that

$$i(\Gamma_{\rm H})\omega_Y = -\mathrm{i}m\,\mathrm{d}\big(K_x^nK_y^{*m}\big).$$

Thus, $\Gamma_{\rm H}$ is the Hamiltonian vector field with respect to ω_Y with $K_x^nK_y^{*m}$ as the Hamiltonian function. Moreover, we can also compute the action of Y on H; a direct calculation gives

$$H_Y \equiv Y(H) = -\mathrm{i}m\big(K_x^nK_y^{*m}\big).$$

To conclude, we have found that the integral of motion J determines the following bi-Hamiltonian system:

$$i(\Gamma_{\rm H})\omega_0 = \mathrm{d}H \qquad i(\Gamma_{\rm H})\omega_Y = \mathrm{d}H_Y.$$

We first remark that $\Gamma_{\rm H}$ is bi-Hamiltonian with respect to two different structures: the canonical symplectic form ω_0 and another one, ω_Y, which is complex. If we write $\omega_Y = \omega_4 + \mathrm{i}\omega_3$, then $\Gamma_{\rm H}$ can be considered as a bi-Hamiltonian system with respect to the following three real forms $(\omega_0, \omega_3, \omega_4)$ (i.e. it is a three-Hamiltonian system). The ω_0-Hamilton equation determined by J,

$$i(X_J)\omega_0 = \mathrm{d}J$$

is also complex; thus it determines two real Hamiltonian equations

$$i(X_4)\omega_0 = \mathrm{d}I_4 \qquad i(X_3)\omega_0 = \mathrm{d}I_3$$

with X_4, X_3, given by $X_J = X_4 + \mathrm{i}X_3$.

As a second remark, the complex 2-form $\Omega = \mathrm{d}K_x \wedge \mathrm{d}K_y^*$ is well defined but it is not symplectic. In fact, it can be proved that $\Omega_1 = \mathrm{Re}(\Omega)$ and $\Omega_2 = \mathrm{Im}(\Omega)$ satisfy

$$\Omega_1 \wedge \Omega_1 = \Omega_2 \wedge \Omega_2 = mn(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}p_x \wedge \mathrm{d}p_y) \qquad \text{and} \qquad \Omega_1 \wedge \Omega_2 = 0$$

so we obtain

$$\Omega \wedge \Omega = (\Omega_1 \wedge \Omega_1 - \Omega_2 \wedge \Omega_2) + 2\mathrm{i}\Omega_1 \wedge \Omega_2 = 0.$$

Thus, the degenerate character of Ω is directly related to its complex nature. Moreover, the kernel of Ω is the distribution generated by Y_{xm} and Y_{yn}^*,

$$\mathrm{Ker}\,\Omega = \{f\,Y_{xm} + gY_{yn}^* \mid f, g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}\}$$

therefore it satisfies

$$[\mathrm{Ker}\,\Omega, \Gamma_{\mathrm{H}}] \subset \mathrm{Ker}\,\Omega.$$

Finally, the 2-form ω_Y is also degenerate. We obtain $\mathrm{Ker}\,\omega_Y = \mathrm{Ker}\,\Omega$, because of the relation between Ω and ω_Y. However, ω_3 and ω_4, defined as $\omega_Y = \omega_4 + \mathrm{i}\omega_3$, are symplectic real forms. Moreover, the form $\omega_0 + \Omega$ is symplectic because of $\{K_x, K_y^*\} = 0$. □

3. Recursion operators

The bi-Hamiltonian structure (ω_0, ω_Y) defines a complex recursion operator R_Y by

$$\omega_Y(X, Y) = \omega_0(R_Y X, Y) \qquad \forall X, Y \in \mathfrak{X}(M)$$

or, equivalently, $R_Y = \widehat{\omega}_0^{-1} \circ \widehat{\omega}_Y$. Since it is complex, it can be written as $R_Y = R_4 + \mathrm{i}R_3$, so that R_4 and R_3 satisfy the relations

$$\omega_3(X, Y) = \omega_0(R_3 X, Y) \quad \text{and} \quad \omega_4(X, Y) = \omega_0(R_4 X, Y).$$

Thus, we have that R_3 and R_4 are given by $R_3 = \widehat{\omega}_0^{-1} \circ \widehat{\omega}_3$ and $R_4 = \widehat{\omega}_0^{-1} \circ \widehat{\omega}_4$.

The important point is that the complex 2-form $\Omega = \mathrm{d}K_x \wedge \mathrm{d}K_y^*$ can be decomposed as $\Omega = \Omega_1 + \mathrm{i}\Omega_2$, where both 2-forms, Ω_1 and Ω_2, are symplectic. Hence, we have, in addition to R_3 and R_4, two other recursion operators R_1 and R_2 associated with the bi-Hamiltonian structures provided by Ω_1 and Ω_2, respectively.

Proposition 3. *The tensor fields R_1 and R_2 are invertible operators which anticommute and satisfy $R_2^2 = R_1^2$.*

Proof. As Ω_1 and Ω_2 are symplectic forms, the operators R_1 and R_2 are invertible. Their coordinate expressions are given by

$$R_1 = \frac{\partial}{\partial y} \otimes \mathrm{d}p_x - \frac{\partial}{\partial x} \otimes \mathrm{d}p_y + mn\left(\frac{\partial}{\partial p_x} \otimes \mathrm{d}y - \frac{\partial}{\partial p_y} \otimes \mathrm{d}x\right) \tag{4}$$

$$R_2 = m\left(\frac{\partial}{\partial y} \otimes \mathrm{d}x + \frac{\partial}{\partial p_x} \otimes \mathrm{d}p_y\right) + n\left(\frac{\partial}{\partial x} \otimes \mathrm{d}y + \frac{\partial}{\partial p_y} \otimes \mathrm{d}p_x\right). \tag{5}$$

Therefore,

$$R_1^2 = mn\,\mathrm{Id} \qquad R_2^2 = mn\,\mathrm{Id}.$$

Moreover, we have

$$R_2 R_1 = n\left(\frac{\partial}{\partial x} \otimes \mathrm{d}p_x - m^2 \frac{\partial}{\partial p_x} \otimes \mathrm{d}x\right) - m\left(\frac{\partial}{\partial y} \otimes \mathrm{d}p_y - n^2 \frac{\partial}{\partial p_y} \otimes \mathrm{d}y\right),$$

and $R_1 R_2 = -R_2 R_1$; therefore $R_2 R_1 + R_1 R_2 = 0$.

We recall that the relation between ω_Y and Ω is $\omega_Y = \omega_4 + \mathrm{i}\omega_3 = -m\mathbb{K}\Omega$, with the complex function $\mathbb{K}$ given by $\mathbb{K} = K_x^{(n-1)} K_y^{*(m-1)} = \mathbb{K}_r + \mathrm{i}\mathbb{K}_i$. Thus, the above two tensor fields, R_3 and R_4, are given by

$$R_4 = -m(\mathbb{K}_r R_1 - \mathbb{K}_i R_2) \tag{6}$$

$$R_3 = -m(\mathbb{K}_r R_2 + \mathbb{K}_i R_1) \tag{7}$$

and, making use of the preceding proposition, we obtain

$$R_4^2 = R_3^2 = r\,\mathrm{Id} \qquad \text{with } r = m^2(mn)^2|\mathbb{K}|^2$$

and

$$R_4 R_3 = m^2|\mathbb{K}|^2 R_1 R_2 \qquad R_3 R_4 = m^2|\mathbb{K}|^2 R_2 R_1$$

where the modulus of $\mathbb{K}$ is a function of the first two integrals, I_1 and I_2,

$$|\mathbb{K}|^2 = \mathbb{K}_r^2 + \mathbb{K}_i^2 = (2I_1)^{(n-1)}(2I_2)^{(m-1)}.$$

Thus, the two tensor fields, R_3 and R_4, anticommute as well. □

Proposition 4. *The complex operator* $R_Y = R_4 + \mathrm{i}R_3$ *is such that* $\mathrm{Image}(R_Y) = \mathrm{Ker}\, R_Y$.

Proof. Let X be a generic vector field on the phase space

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial p_x} + d\frac{\partial}{\partial p_y}$$

with a, b, c, d arbitrary functions. Then, since $R_Y = R_4 + \mathrm{i}R_3$ is given by $R_Y = -m\mathbb{K}(R_1 + \mathrm{i}R_2)$, the subspace Image of R_Y is given by

$$\mathrm{Image}(R_Y) = -\{Z \in \mathfrak{X}(\mathbb{R}^2 \times \mathbb{R}^2) | Z = -m\mathbb{K}(R_1(X) + \mathrm{i}R_2(X))\}.$$

We obtain

$$R_1(X) + \mathrm{i}R_2(X) = -(d - \mathrm{i}nb)Y_{xm} + (c + \mathrm{i}ma)Y_{yn}^*.$$

Thus, $\mathrm{Image}(R_Y)$ is made up of linear combinations of Y_{xm} and Y_{yn}^* with arbitrary complex functions as coefficients. But, since $\mathrm{Ker}\, R_Y = \mathrm{Ker}\,\omega_Y$ and $\mathrm{Ker}\,\omega_Y$ coincides with $\mathrm{Ker}\,\Omega$, which is also spanned by Y_{xm} and Y_{yn}^*, we arrive at $\mathrm{Image}(R_Y) = \mathrm{Ker}\, R_Y$. Consequently $R_Y^2 = R_Y \circ R_Y = 0$.

Given a bi-Hamiltonian system on a manifold M, $i(\Gamma)\omega_0 = \mathrm{d}H_0$ and $i(\Gamma)\omega_1 = \mathrm{d}H_1$, the point is that the tensor field R, that was just defined by the relation between ω_1 and ω_0, induces a sequence of structures. Starting with the basic Hamiltonian system $(\omega_0, \Gamma_0 = \Gamma, \mathrm{d}H_0)$ we can construct a sequence of 2-forms ω_k, of vector fields Γ_k, and of 1-forms α_k, $k = 1, 2, \ldots,$ defined by $\widehat{\omega}_k = \widehat{\omega}_0 \circ R^k$, $\Gamma_k = R^k(\Gamma_0)$, and $\alpha_k = R^k(\mathrm{d}H_0)$. Then it follows that

$$i(\Gamma_0)w_1 = i(\Gamma_1)\omega_0 = \mathrm{d}H_1 \qquad i(\Gamma_0)\omega_2 = i(\Gamma_1)\omega_1 = i(\Gamma_2)\omega_0 = \alpha_2$$

where

$$\begin{array}{ll} \widehat{\omega}_1 = \widehat{\omega}_0 \circ R & \widehat{\omega}_2 = \widehat{\omega}_1 \circ R \\ \Gamma_1 = R(\Gamma_0) & \Gamma_2 = R(\Gamma_1) \\ \mathrm{d}H_1 = R^*(\mathrm{d}H_0) & \alpha_2 = R^*(\mathrm{d}H_1) \end{array}$$

The 1-form α_2 is not necessarily exact, but if there is H_2 such that $\alpha_2 = \mathrm{d}H_2$, then the vector field Γ_1 is a bi-Hamiltonian system as well. An interesting case is when α_2 is not exact but there exist a nonvanishing function F_2 and another function H_2 such that $\alpha_2 = F_2\,\mathrm{d}H_2$. Then F_2^{-1} is an integrating factor for α_2, and the vector field Γ_1 is said to be quasi-bi-Hamiltonian [11, 12].

Coming back to the rational harmonic oscillator as a bi-Hamiltonian system, $i(\Gamma_{\mathrm{H}})\omega_0 = \mathrm{d}H_0$ and $i(\Gamma_{\mathrm{H}})\omega_Y = \mathrm{d}H_Y$, the situation is as follows:

(i) The action of R_Y is such that $\Gamma_{\rm H} \equiv \Gamma_0$ becomes $\Gamma_1 = R_Y(\Gamma_{\rm H}) = -{\rm i}mX_J$,
(ii) ${\rm d}H_0$ transforms into ${\rm d}H_Y = R_Y^*({\rm d}H_0) = -{\rm i}\,{\rm d}J$ and
(iii) ω_0 becomes ω_Y such that $\widehat{\omega}_Y = \widehat{\omega}_0 \circ R_Y$.

We have proved that $R_Y^2 = 0$ because of proposition 4; therefore, Γ_1 transforms into the new field, $\Gamma_2 = R_Y(\Gamma_1) = R_Y^2(\Gamma_{\rm H}) = 0$, while ${\rm d}H_Y$ transforms into $\alpha_2 = R_Y^*({\rm d}H_Y) = R_Y^{*2}({\rm d}H_0) = 0$. Hence, it follows that the equation

$$i(\Gamma_{\rm H})\omega_2 = i(\Gamma_1)\omega_1 = i(\Gamma_2)\omega_0 = \alpha_2$$

becomes

$$i(\Gamma_1)\omega_1 = 0.$$

Note that this last equation corresponds to the property $i(X_J)\omega_Y = 0$.

The harmonic oscillator can be considered as a complex and weakly bi-Hamiltonian system, or alternatively, as endowed with two different real bi-Hamiltonian structures

$$i(\Gamma_{\rm H})\omega_0 = {\rm d}H_0 \qquad i(\Gamma_{\rm H})\omega_4 = m\,{\rm d}I_3 \qquad i(\Gamma_{\rm H})\omega_3 = -m\,{\rm d}I_4.$$

One real structure gives rise to Γ_{13} defined by $\Gamma_{13} = R_3(\Gamma_{\rm H})$, and the other one to $\Gamma_{14} = R_4(\Gamma_{\rm H})$, such that

$$i(\Gamma_{14})\omega_0 = i(\Gamma_{\rm H})\omega_4 = m\,{\rm d}I_3 \qquad i(\Gamma_{13})\omega_0 = i(\Gamma_{\rm H})\omega_3 = -m\,{\rm d}I_4.$$

Moreover, taking into account that $R_4^2 = r\,{\rm Id}$, we obtain

$$\Gamma_{24} = R_4(\Gamma_{14}) = R_4^2(\Gamma_{\rm H}) = r\Gamma_{\rm H}$$
$$\alpha_{24} = R_4^*(m\,{\rm d}I_3) = R_4^{*2}({\rm d}H_0) = r\,{\rm d}H_0$$
$$\widehat{\omega}_{24} = \widehat{\omega}_4 \circ R_4 = \widehat{\omega}_0 \circ R_4^2 = r\widehat{\omega}_0$$

and similar results for R_3. □

Now, making use of all these relations, we can prove the following final proposition concerning the properties of the vector fields X_3 and X_4.

Proposition 5. *Let X_3 and X_4 denote the two infinitesimal canonical symmetries generating the two constants of motion I_3 and I_4. Then X_3 and X_4 are quasi-bi-Hamiltonian systems. Moreover, $\omega_4(X_3, \Gamma_{\rm H}) = \omega_3(X_4, \Gamma_{\rm H}) = 0$.*

Proof. The rational harmonic oscillator is endowed with the two constants of motion I_3 and I_4 which means, via the Hamiltonian–Noether theorem, the existence of two symmetries. They are geometrically represented by two vector fields, X_3 and X_4, that can be uniquely determined as solutions of the following two equations:

$$i(X_3)\omega_0 = {\rm d}I_3 \qquad i(X_4)\omega_0 = {\rm d}I_4.$$

Then we have

$$\Gamma_{13} = R_3(\Gamma_{\rm H}) = -mX_4 \qquad \Gamma_{14} = R_4(\Gamma_{\rm H}) = mX_3.$$

Hence, if we denote by f_{34} the function $f_{34} = m(mn)^2|\mathbb{K}|^2$, we obtain

$$i(X_3)\omega_0 = {\rm d}I_3 \qquad i(X_3)\omega_4 = f_{34}\,{\rm d}H_0$$

and

$$i(X_4)\omega_0 = {\rm d}I_4 \qquad i(X_4)\omega_3 = -f_{34}\,{\rm d}H_0.$$

So, both X_3 and X_4 are quasi-bi-Hamiltonian systems. □

A direct consequence of this property is that the dynamical vector field $\Gamma_{\rm H}$ is orthogonal to X_3 with respect to the symplectic structure ω_4,

$$i(X_3)i(\Gamma_{\rm H})\omega_0 = 0 \quad \text{and} \quad i(X_3)i(\Gamma_{\rm H})\omega_4 = 0.$$

Similarly, we obtain

$$i(X_4)i(\Gamma_{\rm H})\omega_0 = 0 \quad \text{and} \quad i(X_4)i(\Gamma_{\rm H})\omega_3 = 0.$$

Finally, X_3 and X_4 are orthogonal vector fields with respect to both structures, ω_3 and ω_4:

$$i(X_3)i(X_4)\omega_3 = 0 \quad \text{and} \quad i(X_3)i(X_4)\omega_4 = 0.$$

Acknowledgments

This work is dedicated to Professor Domingo González, from the University of Zaragoza, on the occasion of his retirement. Support of Spanish DGI projects, BFM-2000-1066-C03-01 and FPA-2000-1252, is acknowledged.

References

[1] Magri F 1978 *J. Math. Phys.* **19** 1156–62
[2] Marmo G, Morandi G, Simoni A and Ventriglia F 2002 *J. Phys. A: Math. Gen.* **35** 8393–406
[3] Cariñena J F and Ibort L A 1983 *J. Phys. A: Math. Gen.* **16** 1–7
[4] Landi G, Marmo G and Vilasi G 1994 *J. Math. Phys.* **35** 808–15
[5] Nunes da Costa J M and Marle C M 1997 *J. Phys. A: Math. Gen.* **30** 7551–6
[6] Rañada M F 2000 *J. Math. Phys.* **41** 2121–34
[7] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Basel: Birkhauser)
[8] López C, Martínez E and Rañada M F 1999 *J. Phys. A: Math. Gen.* **32** 1241–9
[9] Fradkin D M 1965 *Am. J. Phys.* **33** 207–11
[10] Morandi G, Ferrario C, Lo Vecchio G, Marmo G and Rubano C 1990 *Phys. Rep.* **188** 147–284
[11] Brouzet R, Caboz R, Rabinevo J and Ravoson V 1996 *J. Phys. A: Math. Gen.* **29** 2069–76
[12] Morosi C and Tondo G 1997 *J. Phys. A: Math. Gen.* **30** 2799–806